



TITLE:

# A GENERALIZED SECRETARY PROBLEM WITH UNCERTAIN EMPLOYMENT(Mathematical Structure of Optimization Theory)

AUTHOR(S):

Tamaki, Mitsushi

---

CITATION:

Tamaki, Mitsushi. A GENERALIZED SECRETARY PROBLEM WITH UNCERTAIN EMPLOYMENT(Mathematical Structure of Optimization Theory). 数理解析研究所講究録 1994, 864: 154-163

ISSUE DATE:

1994-04

URL:

<http://hdl.handle.net/2433/83895>

RIGHT:

# A GENERALIZED SECRETARY PROBLEM WITH UNCERTAIN EMPLOYMENT

愛知大学 玉置光司(Mitsushi Tamaki)

## 1. Introduction

We shall consider and solve a modification of the classical secretary problem with uncertain employment. The problem is described as follows :  $n$  applicants appear one by one in random order with all  $n!$  permutations equally likely. We are able, at any time, to rank the applicants that have so far appeared. As each applicant appears we must decide whether or not to make an offer to that applicant with the objective of maximizing the probability of selecting the best(most preferred)applicant. It is assumed that each applicant only accepts an offer of employment with constant probability  $p$  and that an applicant to whom an offer is not made cannot be recalled later (see Smith[1] and Tamaki[2]).

The problem we consider here allows the applicants to refuse an offer depending on the rank of the applicant. Let  $P_i(Q_i = 1 - P_i)$  be the acceptance probability (rejection probability) of the  $i$ -th ranked applicant  $1 \leq i \leq n$ . We treat the best choice problem in Section 2 and the Gusein-Zade problem in Section 3.

## 2. Best Choice Problem

Our objective is to maximize the probability of choosing the best applicant. Imagine a situation where  $r-1$  applicants have so far appeared, and offer(s) were made to  $k$  of them but rejected  $0 \leq k \leq r-1 \leq n$ . Then, if  $k \geq 1$ , this state is described as  $(r-1; i_1, i_2, \dots, i_k)$ , where the information pattern  $(i_1, i_2, \dots, i_k)$  represents the relative ranks, among the first  $r-1$  applicants, of those who have rejected offers arranged in ascending order, i.e.,  $1 \leq i_1 < i_2 < \dots < i_k \leq r-1$ . Furthermore denote by  $(r; i | i_1, i_2, \dots, i_k)$ ,  $1 \leq i \leq r$  the state where, after leaving state  $(r-1; i_1, i_2, \dots, i_k)$ , we have just observed the  $r$ -th applicant to be relatively  $i$ -th best. When  $k=0$ , i.e., no offer has been made so far, the information pattern is denoted by  $\Phi$  and the corresponding states will be denoted by  $(r-1; \Phi)$  and  $(r; i | \Phi)$ . In this section, our trial is said to be a success if the chosen applicant is the overall best.

Let  $v_{r-1}(i_1, i_2, \dots, i_k)$  be the probability of success assuming that we proceed optimally after leaving state  $(r-1; i_1, i_2, \dots, i_k)$ . Also let  $s_r(i | i_1, i_2, \dots, i_k)$  ( $c_r(i | i_1, i_2, \dots, i_k)$ ) be the corresponding probability when we make (when we do not make) an offer to the  $r$ -th applicant in state  $(r; i | i_1, i_2, \dots, i_k)$  and proceed optimally thereafter. Corresponding to states  $(r-1; \Phi)$  and  $(r; i | \Phi)$ ,  $v_{r-1}(\Phi)$  and  $s_r(i | \Phi)$ ,  $c_r(i | \Phi)$  can be respectively defined in a similar way. Let  $p_r(i | i_1, i_2, \dots, i_k)$  be the transition probability from state  $(r-1; i_1, i_2, \dots, i_k)$  into state  $(r; i | i_1, i_2, \dots, i_k)$ . Also let  $p_r(i | \Phi)$  be the transition probability from state  $(r-1; \Phi)$  into state  $(r; i | \Phi)$ . Then we have from the principle of optimality,

$$v_{r-1}(i_1, \dots, i_k) = \sum_{i=1}^r p_r(i | i_1, \dots, i_k) \cdot \max\{s_r(i | i_1, \dots, i_k), c_r(i | i_1, \dots, i_k)\} \quad (1 \leq k \leq r-1 < n) \quad (2.1)$$

$$v_{r-1}(\Phi) = \sum_{i=1}^r p_r(i | \Phi) \cdot \max\{s_r(i | \Phi), c_r(i | \Phi)\} \quad (0 \leq r-1 < n) \quad (2.2)$$

with the boundary conditions  $v_n(i_1, \dots, i_k) = 0$  and  $v_n(\phi) = 0$ .

Obviously optimal success probability will be calculated as  $v_0(\phi)$ . It is easy to see

$$Pr(i | \phi) = 1/r, \quad 1 \leq i \leq r \quad (2.3)$$

from the assumption that the arrival orders of the applicants are equally likely. However,  $Pr(i | i_1, i_2, \dots, i_k)$  is not equal to  $1/r$  in general, because the information pattern  $(i_1, i_2, \dots, i_k)$  observed so far has influence on estimating the future arrival of the remaining applicants.

To derive  $Pr(i | i_1, i_2, \dots, i_k)$ , some notations must be introduced. For convenience of exposition, we denote by  $C_i^r$ ,  $1 \leq i \leq r$ ,  $1 \leq r \leq n$  the  $i$ -th best among the first  $r$  applicants (in particular,  $C_i^n$  represents the applicant of absolute rank  $i$ ).

Let  $A(r, i; n)$  be a random variable representing the absolute rank of the applicant  $C_i^r$ , i.e.,  $A(r, i; n) = j$  if  $C_i^r$  is  $C_j^n$ . Then it is easy to see that the joint probability  $P(A(r, i_1; n) = j_1, A(r, i_2; n) = j_2, \dots, A(r, i_k; n) = j_k)$ , which is denoted simply by  $P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r)$ , is given by

$$P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r) = \frac{\binom{j_1-1}{i_1-1} \binom{j_2-j_1-1}{i_2-i_1-1} \cdots \binom{j_k-j_{k-1}-1}{i_k-i_{k-1}-1} \binom{n-j_k}{r-i_k}}{\binom{n}{r}} \quad (2.4)$$

for  $(j_1, j_2, \dots, j_k) \in W_r(i_1, i_2, \dots, i_k)$ ,

where  $W_r(i_1, i_2, \dots, i_k)$  stands for the set of possible values  $(j_1, j_2, \dots, j_k)$  for given values  $(i_1, i_2, \dots, i_k)$ , i.e.,  $W_r(i_1, i_2, \dots, i_k) = \{(j_1, j_2, \dots, j_k) : j_1 < j_2 < \dots < j_k, i_s \leq j_s \leq n-r+i_s, 1 \leq s \leq k\}$ .

Some properties of  $P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r)$  are listed in the following lemma.

#### LEMMA 2.1

- (i)  $P(j_1, j_2, \dots, j_k; n | i_1+1, i_2+1, \dots, i_k+1; r)$
- $$= \left(\frac{r}{n-r+1}\right) \left(\frac{j_1-i_1}{i_1}\right) P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r-1)$$
- $P(j_1, \dots, j_k; n | i_1, \dots, i_{s-1}, i_s+1, \dots, i_k+1; r)$
- $$= \left(\frac{r}{n-r+1}\right) \left(\frac{j_s-i_s-(j_{s-1}-i_{s-1})}{i_s-i_{s-1}}\right) P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r-1) \quad (2 \leq s \leq k)$$
- $P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r)$
- $$= \left(\frac{r}{n-r+1}\right) \left(\frac{(n-r+1)-(j_k-i_k)}{r-i_k}\right) P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r-1)$$
- (ii)  $P(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r)$
- $$= P(j_1, j_2, \dots, j_{s-1}, j_s-1 | i_1, i_2, \dots, i_{s-1}, i_s-1) P(j_s, j_{s+1}, \dots, j_k; n | i_s, i_{s+1}, \dots, i_k; r) \quad (2 \leq s \leq k)$$
- (iii)  $P(j_1, j_2, \dots, j_k; n | i_1+1, i_2+1, \dots, i_k+1; r)$

$$= \left(\frac{r}{n}\right) \left(\frac{j_1-1}{i_1}\right) p(j_1-1, j_2-1, \dots, j_k-1, n-1 \mid i_1, i_2, \dots, i_k; r-1)$$

$$\begin{aligned} \text{(iv)} \quad & p(j_1, \dots, j_k; n \mid i_1, \dots, i_k; r-1) \\ &= \left(\frac{i_1}{r}\right) p(j_1, \dots, j_k; n \mid i_1+1, \dots, i_k+1; r) \\ &+ \sum_{s=2}^k \left(\frac{i_s-i_{s-1}}{r}\right) p(j_1, \dots, j_k; n \mid i_1, \dots, i_{s-1}, i_s+1, \dots, i_k+1; r) \\ &+ \left(\frac{r-i_k}{r}\right) p(j_1, \dots, j_k; n \mid i_1, \dots, i_k; r) \end{aligned}$$

PROOF. Straightforward from (2.4).

Define, for  $1 \leq i_1 < i_2 < \dots < i_k \leq r$ ,  $1 \leq r \leq n$

$$a_r(i_1, i_2, \dots, i_k) = E\left[\prod_{t=1}^k q_{A(r, i_t; n)}\right], \quad (2.5)$$

where  $E$  denotes an operator of taking expectation.  $a_r(i_1, i_2, \dots, i_k)$  implies the probability that all  $k$  offers will be rejected, provided that these offers are given to  $C_{i_1}^r, C_{i_2}^r, \dots, C_{i_k}^r$  (we write  $a_r(i_1, i_2, \dots, i_k; n)$  to make explicit the dependence on  $n$ , total number of applicants that appears, if necessary). We can write (2.5) as

$$a_r(i_1, i_2, \dots, i_k) = \sum \sum \dots \sum \left(\prod_{t=1}^k q_{j_t}\right) p(j_1, j_2, \dots, j_k; n \mid i_1, i_2, \dots, i_k; r)$$

where summations with respect to  $(j_1, j_2, \dots, j_k)$  are taken over  $W_r(i_1, i_2, \dots, i_k)$ .

We now have the following lemma.

LEMMA 2.2

$$p_r(i \mid i_1, i_2, \dots, i_k) = \begin{cases} \left(\frac{1}{r}\right) \left[\frac{a_r(i_1+1, i_2+1, \dots, i_k+1)}{a_{r-1}(i_1, i_2, \dots, i_k)}\right], & 1 \leq i \leq i_1 \\ \left(\frac{1}{r}\right) \left[\frac{a_r(i_1, \dots, i_{s-1}, i_s+1, \dots, i_k+1)}{a_{r-1}(i_1, i_2, \dots, i_k)}\right], & i_{s-1} < i \leq i_s (2 \leq s \leq k) \\ \left(\frac{1}{r}\right) \left[\frac{a_r(i_1, i_2, \dots, i_k)}{a_{r-1}(i_1, i_2, \dots, i_k)}\right], & i_k < i \leq r \end{cases}$$

PROOF. Let  $\tilde{p}(j_1, j_2, \dots, j_k; n \mid i_1, i_2, \dots, i_k; r-1)$  be the conditional joint probability that  $C_{i_t}^{r-1}$  is in effect  $C_{j_t}^n$ ,  $1 \leq t \leq k$  provided that offers given to  $C_{i_1}^{r-1}, \dots, C_{i_k}^{r-1}$  are all rejected. Then by the Bayes formula

$$\begin{aligned} \tilde{p}(j_1, j_2, \dots, j_k; n \mid i_1, i_2, \dots, i_k; r-1) \\ = \frac{(q_{j_1} q_{j_2} \dots q_{j_k}) p(j_1, j_2, \dots, j_k; n \mid i_1, i_2, \dots, i_k; r-1)}{a_{r-1}(i_1, i_2, \dots, i_k)} \end{aligned} \quad (2.6)$$

Let  $R$  be the relative rank of the  $r$ -th applicant. We easily see that the conditional probability distribution of  $R$ , given that  $C_{i_t}^{r-1}$  is  $C_{j_t}^n$  for  $1 \leq t \leq k$ , is given by

$$P(R=i | j_1, j_2, \dots, j_k) = \begin{cases} \left(\frac{1}{i_1}\right) \left(\frac{j_1 - i_1}{n - r + 1}\right) & 1 \leq i \leq i_1 \\ \left(\frac{1}{i_s - i_{s-1}}\right) \left(\frac{(j_s - i_s) - (j_{s-1} - i_{s-1})}{n - r + 1}\right) & i_{s-1} < i \leq i_s \quad (2 \leq s \leq k) \\ \left(\frac{1}{r - i_k}\right) \left(\frac{(n - r + 1) - (j_k - i_k)}{n - r + 1}\right) & i_k < i \leq r \end{cases} \quad (2.7)$$

• Thus the result follows from (2.6), (2.7) and Lemma 2.1(i), since  $Pr(i | i_1, i_2, \dots, i_k)$  is calculated through

$$p_r(i | i_1, i_2, \dots, i_k) = \sum \sum \dots \sum P(R=i | j_1, j_2, \dots, j_k) \cdot \tilde{p}(j_1, j_2, \dots, j_k; n | i_1, i_2, \dots, i_k; r-1),$$

where summations are taken over  $W_{r-1}(i_1, i_2, \dots, i_k)$ .

Some properties of  $a_r(i_1, i_2, \dots, i_k)$  are listed in the following lemma.

LEMMA 2.3

(i) For  $1 < r \leq n$ ,

$$\begin{aligned} a_{r-1}(i_1, i_2, \dots, i_k) &= \left(\frac{i_1}{r}\right) a_r(i_1+1, i_2+1, \dots, i_k+1) \\ &+ \sum_{s=2}^k \left(\frac{i_s - i_{s-1}}{r}\right) a_r(i_1, \dots, i_{s-1}, i_s+1, \dots, i_k+1) \\ &+ \left(\frac{r - i_k}{r}\right) a_r(i_1, i_2, \dots, i_k) \end{aligned}$$

(ii)

$$\begin{aligned} a_r(i_1, i_2, \dots, i_k; n) \\ = \sum \sum \dots \sum a_{i_r-1}(i_1, \dots, i_{s-1}, j_s-1) \left(\prod_{t=s}^k q_{j_t}\right) P(j_s, \dots, j_k; n | i_s, \dots, i_k; r) \end{aligned}$$

where summations with respect to  $(j_s, \dots, j_k)$  are taken over  $W_r(i_s, \dots, i_k)$ ,  $(2 \leq s \leq k)$ .

(iii) Assume that  $\{q_j\}$  is non-increasing in  $j$ . Then  $a_r(i_1, i_2, \dots, i_k)$  is non-decreasing in  $r$  and non-increasing in  $i_s$ .

PROOF. (i) is immediate from Lemma 2.2, since  $\sum_{i=1}^r Pr(i | i_1, i_2, \dots, i_k)$  must be unity.

(ii) is straightforward from Lemma 2.1(ii).

(iii) can be shown by induction on  $r$ .

In our problem, the  $j$ -th best applicant is assumed to reject an offer with probability  $q_j$ . So we denote this problem by  $\begin{bmatrix} 1 & 2 & \dots & n \\ q_1 & q_2 & \dots & q_n \end{bmatrix}$ . Consider a modified problem  $\begin{bmatrix} 1 & 2 & \dots & n-1 \\ q_2 & q_3 & \dots & q_n \end{bmatrix}$ , where total number of applicants is  $n-1$  and the  $j$ -th best applicant rejects with probability  $q_{j+1}$ ,  $1 \leq j \leq n-1$  and let  $b_r(i_1, i_2, \dots, i_k)$ ,  $1 \leq r \leq n-1$  denote the probability that all  $k$  offers will be rejected when these offers are given to  $C_{i_1}^r, C_{i_2}^r, \dots, C_{i_k}^r$ . More specifically

$$b_r(i_1, i_2, \dots, i_k) = \sum \sum \dots \sum \left(\prod_{t=1}^k q_{j_t+1}\right) P(j_1, j_2, \dots, j_k; n-1 | i_1, i_2, \dots, i_k; r) \quad (2.8)$$

Note that, since  $b_r(i_1, i_2, \dots, i_k)$  corresponds to  $a_r(i_1, i_2, \dots, i_k)$  in the original problem, Lemma 2.3(i) holds with  $a_r(i_1, i_2, \dots, i_k)$  replaced by  $b_r(i_1, i_2, \dots, i_k)$  for  $2 \leq r \leq n-1$  and  $b_r(i_1, i_2, \dots, i_k)$

can be solved recursively starting with the boundary condition  $b_{n-1}(i_1, i_2, \dots, i_k) = \prod_{t=1}^k q_{i_t+1}$

We can now express  $s_r(i | i_1, \dots, i_k)$  and  $c_r(i | i_1, \dots, i_k)$  in terms of  $v_r(i_1, \dots, i_k)$ ,  $a_r(i_1, i_2, \dots, i_k)$  and  $b_r(i_1, i_2, \dots, i_k)$ .

#### LEMMA 2.4

(i) For  $1 < r \leq n$

$$s_r(i | i_1, \dots, i_k) =$$

$$\begin{cases} p_1\left(\frac{r}{n}\right) \left[ \frac{b_{r-1}(i_1, i_2, \dots, i_k)}{a_r(i_1+1, i_2+1, \dots, i_k+1)} \right] + v_r(1, i_1+1, \dots, i_k+1) \left[ \frac{a_r(1, i_1+1, \dots, i_k+1)}{a_r(i_1+1, i_2+1, \dots, i_k+1)} \right], & i=1 \\ v_r(i, i_1+1, \dots, i_k+1) \left[ \frac{a_r(i, i_1+1, \dots, i_k+1)}{a_r(i_1+1, i_2+1, \dots, i_k+1)} \right], & 1 < i \leq i_1 \\ v_r(i_1, \dots, i_{s-1}, i, i_s+1, \dots, i_k+1) \left[ \frac{a_r(i_1, \dots, i_{s-1}, i, i_s+1, \dots, i_k+1)}{a_r(i_1, \dots, i_{s-1}, i_s+1, \dots, i_k+1)} \right], & i_{s-1} < i \leq i_s \\ & (2 \leq s \leq k) \\ v_r(i_1, \dots, i_k, i) \left[ \frac{a_r(i_1, \dots, i_k, i)}{a_r(i_1, \dots, i_k)} \right], & i_k < i \leq r \end{cases}$$

$$c_r(i | i_1, \dots, i_k) = \begin{cases} v_r(i_1+1, \dots, i_k+1), & 1 \leq i \leq i_1 \\ v_r(i_1, \dots, i_{s-1}, i_s+1, \dots, i_k+1), & i_{s-1} < i \leq i_s \\ & (2 \leq s \leq k) \\ v_r(i_1, \dots, i_k), & i_k < i \leq r \end{cases}$$

(ii) For  $1 \leq r \leq n$ ,

$$s_r(i | \phi) = \begin{cases} p_1\left(\frac{r}{n}\right) + a_r(1)v_r(1), & i=1 \\ a_r(i)v_r(i), & 1 < i \leq r \end{cases}$$

$$c_r(i|\phi) = v_r(\phi), \quad 1 \leq r$$

PROOF. We'll only derive  $s_r(1|i_1, \dots, i_k)$ , since others can be obtained in a similar way. Suppose that we are in state  $(r, 1|i_1, \dots, i_k)$ , the forecasting probability that  $C_{i+1}^r$  is  $C_{j_i}^n$  for  $1 \leq i \leq k$  is given by  $\tilde{p}(j_1, j_2, \dots, j_k; n|i_1+1, i_2+1, \dots, i_k+1; r)$  defined in (2.6). On the other hand, given that  $C_{i+1}^r$  is  $C_{j_i}^n$ , making an offer to  $C_i^r$  leads to a success with probability

$$p_1 p(1|j_1-1|1|i_1) + a_{i_1}(1|j_1-1)v_r(1, i_1+1, \dots, i_k+1)$$

The first term corresponds to acceptance of the offer and the second term corresponds to rejection and subsequent continuation in an optimal manner. Thus we have

$$\begin{aligned} s_r(1|i_1, \dots, i_k) &= \sum \dots \sum [p_1 p(1|j_1-1|1|i_1) + a_{i_1}(1|j_1-1)v_r(1, i_1+1, \dots, i_k+1)] \\ &\quad \times \tilde{p}(j_1, j_2, \dots, j_k; n|i_1+1, i_2+1, \dots, i_k+1; r), \end{aligned} \quad (2.9)$$

where summations with respect to  $(j_1, j_2, \dots, j_k)$  are taken over  $W_r(i_1+1, \dots, i_k+1)$ . From Lemma 2.1(iii), the first term in the RHS of (2.9) can be reduced to

$$\begin{aligned} &\frac{p_1}{a_r(i_1+1, \dots, i_k+1)} \sum \dots \sum \left( \prod_{t=1}^k q_{j_t} \right) \left( \frac{i_1}{j_1-1} \right) p(j_1, \dots, j_k; n|i_1+1, \dots, i_k+1; r) \\ &= p_1 \left( \frac{r}{n} \right) \frac{1}{a_r(i_1+1, \dots, i_k+1)} \sum \dots \sum \left( \prod_{t=1}^k q_{j_t} \right) p(j_1-1, \dots, j_k-1; n-1|i_1, \dots, i_k; r-1) \\ &= p_1 \left( \frac{r}{n} \right) \frac{b_{r-1}(i_1, \dots, i_k)}{a_r(i_1+1, \dots, i_k+1)} \end{aligned} \quad (2.10)$$

The second term can be written as, from Lemma 2.3(ii),

$$\begin{aligned} &\frac{v_r(1, i_1+1, \dots, i_k+1)}{a_r(i_1+1, \dots, i_k+1)} \sum \dots \sum a_{i_1}(1|j_1-1) \left( \prod_{t=1}^k q_{j_t} \right) p(j_1, \dots, j_k; n|i_1+1, \dots, i_k+1; r) \\ &= v_r(1, i_1+1, \dots, i_k+1) \frac{a_r(1, i_1+1, \dots, i_k+1)}{a_r(i_1+1, \dots, i_k+1)} \end{aligned} \quad (2.11)$$

Substituting (2.10) and (2.11) into (2.9) yields the desired result.

$c_r(i|i_1, i_2, \dots, i_k)$ ,  $i_{s-1} < i \leq i_s$ , is immediate since, if we do not make an offer, the information pattern is changed by incrementing  $i_t$  by one for  $t \geq s$  (when  $i \leq i_1$ , each component of the information pattern increases by one, and when  $i > i_k$ , no change occurs). (ii) is easy to see and hence omitted.

Define

$$\begin{aligned} V_r(i_1, \dots, i_k) &= a_r(i_1, \dots, i_k) v_r(i_1, \dots, i_k), \quad 1 \leq r \leq n \\ V_r(\phi) &= v_r(\phi), \quad 0 \leq r \leq n \end{aligned}$$

and apply Lemmas 2.2 and 2.4 to (2.1) and (2.2). Then we have the following lemma.

LEMMA 2.5

Given additional information,  $V_r(i_1, \dots, i_k)$  does not increase. To be more precise, for information pattern  $(i_1, \dots, i_k)$  with  $i_s - i_{s-1} > 1$  for some  $s$  ( $2 \leq s \leq k$ ),

$$V_r(i_1, \dots, i_{s-1}, i_s, \dots, i_k) \geq V_r(i_1, \dots, i_{s-1}, i, i_s, \dots, i_k) \quad (2.12)$$

$i_{s-1} < i < i_s$ .

When  $i < i_1$  or  $i > i_k$ ,  $V_r(i_1, \dots, i_k) \geq V_r(i, i_1, \dots, i_k)$  or  $V_r(i_1, \dots, i_k) \geq V_r(i_1, \dots, i_k, i)$  holds respectively. Moreover  $V_r(\phi) \geq V_r(i)$  for  $1 \leq i \leq r$ .  
Thus

$$\begin{aligned} V_{r-1}(i_1, \dots, i_k) &= \frac{1}{r} \max \{ p_1 \left( \frac{r}{n} \right) b_{r-1}(i_1, \dots, i_k) + V_r(1, i_1+1, \dots, i_k+1), V_r(i_1+1, \dots, i_k+1) \} \\ &\quad + \left( \frac{i_1-1}{r} \right) V_r(i_1+1, \dots, i_k+1) \\ &\quad + \sum_{t=2}^k \left( \frac{i_t-i_{t-1}}{r} \right) V_r(i_1, \dots, i_{t-1}, i_t+1, \dots, i_k+1) \\ &\quad + \left( \frac{r-i_k}{r} \right) V_r(i_1, \dots, i_k) \end{aligned} \quad (2.13)$$

$(1 < r \leq n; V_n(i_1, \dots, i_k) = 0)$

$$V_{r-1}(\phi) = \frac{1}{r} \max \{ p_1 \left( \frac{r}{n} \right) + V_r(1), V_r(\phi) \} + \left( \frac{r-1}{r} \right) V_r(\phi) \quad (2.14)$$

$(1 \leq r \leq n; V_n(\phi) = 0)$

PROOF. We show (2.12) by induction on  $r$ . For  $r=n-1$ , (2.12) is evident since

$$V_{n-1}(i_1, \dots, i_k) = \left( \frac{p_1}{n} \right) b_{n-1}(i_1, \dots, i_k) = \left( \frac{p_1}{n} \right) \prod_{t=1}^k q_{i_t+1} \quad (2.15)$$

Assume that (2.12) holds. Then (2.13) holds and yields immediately

$$V_{r-1}(i_1, \dots, i_{s-1}, i_s, \dots, i_k) - V_{r-1}(i_1, \dots, i_{s-1}, i, i_s, \dots, i_k) \geq 0$$

from the induction hypothesis and the fact that  $b_{r-1}(i_1, \dots, i_k)$ , by definition, does not increase with additional information.

The following lemma is concerned with the asymptotic result.

LEMMA 2.6

Let  $n$  and  $r$  tend to infinity with  $r/n=x$ , then  $V_r(i_1, \dots, i_k)$  approaches  $V(x | i_1, \dots, i_k)$ , which satisfies the following differential equations:



$$\begin{aligned}
& -\frac{d}{dx}V(x | i_1, \dots, i_k) \\
& = \frac{1}{x} \max\{p_1 x b(x | i_1, \dots, i_k) + V(x | 1, i_1+1, \dots, i_k+1), V(x | i_1+1, \dots, i_k+1)\} \\
& \quad + \left(\frac{i_1-1}{x}\right)V(x | i_1+1, \dots, i_k+1) \\
& \quad + \sum_{t=2}^k \left(\frac{i_t-i_{t-1}}{x}\right)V(x | i_1, \dots, i_{t-1}, i_t+1, \dots, i_k+1) \\
& \quad - \left(\frac{i_k}{x}\right)V(x | i_1, \dots, i_k),
\end{aligned}$$

$$-\frac{d}{dx}V(x|\phi) = \frac{1}{x} \max\{p_1 x + V(x|1), V(x|\phi)\} - \frac{1}{x}V(x|\phi),$$

where

$$\begin{aligned}
b(x | i_1, \dots, i_k) &= \sum_{j_1=i_1}^{\infty} \sum_{j_2=j_1+i_2-i_1}^{\infty} \dots \sum_{j_k=j_{k-1}+i_k-i_{k-1}}^{\infty} \\
&\quad \left(\prod_{s=1}^k q_{j_s+1}\right) \binom{j_1-1}{i_1-1} \binom{j_2-j_1-1}{i_2-i_1-1} \dots \binom{j_k-j_{k-1}-1}{i_k-i_{k-1}-1} x^{i_k} (1-x)^{j_k-i_k}
\end{aligned}$$

PROOF. immediate

EXAMPLE 2.1

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \dots \\ q_1 & q_2 & q_3 & 0 & 0 & \dots \end{bmatrix}$$

An optimal policy is threshold type with critical number  $\alpha$  i.e., pass over the first  $\alpha$  applicants and then give an offer successively to a candidate that appears until an offer is accepted or deadend comes.

$\alpha$  is the unique root  $x$  of the equation

$$1 + 2[q_2 + q_3(1+q_2)](1-x) - \frac{3}{4}q_3(1+q_2)(1-x)^2 = -(1+q_2)\left(1 + \frac{1}{2}q_3\right)\log x$$

Moreover the optimal success probability is

$$P(S) = p_1 \alpha \left[ (1+q_2)\left(1 + \frac{1}{2}q_3\right) - (q_2 + q_3 + q_2 q_3) \alpha + \frac{1}{2}q_3(1+q_2)\alpha^2 \right]$$

EXAMPLE 2.2

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots \\ q_1 & q_2 & q & q & \dots \end{bmatrix}$$

This problem is simplified by noting that

$$V_i(i_1, \dots, i_k) = \begin{cases} q^{k-1} V_i(2), & \text{if } i_1 \neq 1 \\ q^{k-1} V_i(1), & \text{if } i_1 = 1 \end{cases}$$

An optimal policy is threshold type with critical number  $\alpha$ , where  $\alpha$  is the unique root  $x$  of the equation

$$(1+q)(1-q+q_2)x = (1-q)(1+q_2)x^{1-q} + 2q(q_2-q)x^2$$

The optimal success probability is

$$P(S) = \frac{p_1}{q(1+q)} [(1+q_2)\alpha^{1-q}(1+q)(1-q+q_2)\alpha + q(q_2-q)\alpha^2]$$

### EXAMPLE 2.3

$$\begin{bmatrix} 1 & 2 & \dots & m-1 & m & m+1 & \dots \\ q_1 & 0 & \dots & 0 & q_m & 0 & \dots \end{bmatrix}$$

An optimal policy is not necessarily threshold type. Consider the case  $\begin{bmatrix} 1 & 2 & \dots & m-1 & m & m+1 & \dots \\ q_1 & 0 & \dots & 0 & q_m & 0 & \dots \end{bmatrix}$  with  $m$  sufficiently large. Then if the first offer is rejected, we'll pass over about  $\epsilon^m$  candidates and then give an offer. My conjecture is that, if  $j_j$  is non-increasing in  $j$ , then the optimal policy is threshold type.

### 3. Gusein-Zade Problem

Our objective is to maximize the probability of choosing either the best or the second best. Corresponding to  $a_r(i_1, i_2, \dots, i_k)$  and  $b_r(i_1, i_2, \dots, i_k)$ , define  $c_r(i_1, i_2, \dots, i_k)$  for the modified problem  $\begin{bmatrix} 1 & 2 & \dots & n-2 \\ q_3 & q_4 & \dots & q_n \end{bmatrix}$ . Then we have the following optimality equations.

$$\begin{aligned} V_{r-1}(i_1, i_2, \dots, i_k) &= \frac{1}{r} \max \left\{ p_1 \left( \frac{r}{n} \right) b_{r-1}(i_1, i_2, \dots, i_k) + p_2 \frac{r(n-r)}{n(n-1)} c_{r-1}(i_1, i_2, \dots, i_k) \right. \\ &\quad \left. + V_r(1, i_1+1, \dots, i_k+1), V_r(i_1+1, \dots, i_k+1) \right\} \\ &\quad + \frac{1}{r} \max \left\{ p_2 \left( \frac{r-1}{n-1} \right) c_{r-2}(i_1-1, i_2-1, \dots, i_k-1) \right. \\ &\quad \left. + V_r(2, i_1+1, \dots, i_k+1), V_r(i_1+1, \dots, i_k+1) \right\} \\ &\quad + \left( \frac{i_1-2}{r} \right) V_r(i_1+1, \dots, i_k+1) \\ &\quad + \sum_{t=2}^k \left( \frac{i_t-i_{t-1}}{r} \right) V_r(i_1, \dots, i_{t-1}, i_t+1, \dots, i_k+1) \\ &\quad + \left( \frac{r-i_k}{r} \right) V_r(i_1, \dots, i_k), \end{aligned}$$

$$\begin{aligned} V_{r-1}(\phi) &= \frac{1}{r} \max \left\{ p_1 \left( \frac{r}{n} \right) + p_2 \frac{r(n-r)}{n(n-1)} + V_r(1), V_r(\phi) \right\} \\ &\quad + \frac{1}{r} \max \left\{ p_2 \frac{r-1}{n-1} + V_r(2), V_r(\phi) \right\} \\ &\quad + \left( 1 - \frac{2}{r} \right) V_r(\phi). \end{aligned}$$

Letting  $n$  and  $r$  tend to infinity, we can derive the differential equations analogous to Lemma 2.7.

### EXAMPLE 3.1

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ q_1 & q_2 & 0 & 0 \end{bmatrix}$$

An optimal policy is described in terms of  $d_1$ ,  $d_2$  and  $s_2$ . As for the first offer, give an offer to relatively best if he appears after  $d_1$  and give an offer to relatively second best if he

appears after  $d_2$  ( $d_1 < d_2$ ). If the first offer was given to relatively best but rejected then we immediately give the second offer to the next relatively best but give the second offer to the relatively second best only when he appears after  $s_2$ .

Parameter space is partitioned into  $R_1$  and  $R_2$ , such that

$$R_1 = \{(p_1, p_2) : 0 < p_2 \leq p_2^*(p_1), 0 < p_1 \leq 1\}$$

$$R_2 = \{(p_1, p_2) : p_2^*(p_1) < p_2 \leq 1, 0 < p_1 \leq 1\}$$

where  $p_2^*(p_1)$  is defined, for a given  $p_1$ , as a unique root  $p_2$  of the equation

$$e^{-\delta} = \frac{2(p_1 + p_2)}{2(p_1 + 2p_2) - (1 - p_1)p_2\delta}$$

and  $\delta$  is defined as

$$\delta = \frac{q_1 p_2}{p_1 q_2 + q_1 p_2}$$

It can be shown that

$$d_1 \leq s_2 \leq d_2, \quad \text{if } (p_1, p_2) \in R_1$$

$$d_1 \leq d_2 \leq s_2, \quad \text{if } (p_1, p_2) \in R_2$$

For  $(p_1, p_2) \in R_1$ ,  $d_1$ ,  $d_2$  and  $s_2$  are defined as follows:

$$s_2 = \exp(-\delta)$$

$d_2$  is a unique root  $x$  of the equation

$$(p_1 + p_2) + \frac{(p_1 q_2 + q_1 p_2)x}{2} \log^2 x = (p_1 + 2p_2)x$$

$d_1$  is a unique root  $x$  of the equation

$$2[p_2 + a(1 - \delta)]x - (p_1 + p_2 + as_2) \log x = (p_1 + p_2)(1 - \log d_2) + a[(3 + \delta)s_2 - (1 - \log d_2)d_2]$$

where  $a = p_1 q_2 + q_1 p_2$ .

Moreover

$$P(S) = d_1[(p_1 + p_2 + as_2) - \{p_2 + a(1 - \delta)\}d_1]$$

## References

- [1] Smith, M.H. 1975. A Secretary Problem With Uncertain Employment. J. Appl. Prob. 12, 620-624.
- [2] Tamaki, M. 1991. A Secretary Problem With Uncertain Employment and Best Choice of Available Candidates. Operat. Res. 39, 274-284.